

# Union of $n$ Disks: Remote Centers, Common Origin

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November 16, 2015

**ABSTRACT.** Explicit area expressions are known for a special case, due to Tao & Wu (1987), and lead to calculation of integrals in applied probability.

A collection of  $n$  planar disks is said to have **remote centers** if the  $i^{\text{th}}$  disk never contains the  $j^{\text{th}}$  center, for any  $1 \leq i \neq j \leq n$ . It has **common origin** if the point  $\vec{0}$  is on the boundary of each disk. No further assumptions are made concerning the relative sizes of the disks or the extent to which they overlap. Let the disks be centered at  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ . It follows that  $\vec{0}$  is closer to  $\vec{r}_j$  than any other  $\vec{r}_i$ . Let  $r_j$  denote the length of  $\vec{r}_j$  (the  $j^{\text{th}}$  radius) and  $r_{i,j}$  denote the length of  $\vec{r}_i - \vec{r}_j$ . The constraints  $r_j < r_{i,j}$  are crucial for the calculation of certain integrals in [1], which in turn give the probability that an individual survives a random type of violent shootout.

Consider the integral

$$c_n = \frac{1}{n!} \int_{r_j < r_{i,j}} \exp[-V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)] d\vec{r}_1 d\vec{r}_2 \dots d\vec{r}_n$$

where  $V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$  is the area of the union of  $n$  disks centered at  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  and intersecting at  $\vec{0}$ . The value of  $V$  can be found symbolically by use of RegionUnion and RegionMeasure functions in MATHEMATICA 10. Such technology does not currently allow us to evaluate  $c_n$  directly because computer memory is quickly exhausted in the required numerical quadrature. One aim of this paper is to revisit formulas in [1], focusing on an elaborate change of variables and leading to a less-intensive indirect calculation. The notation employed previously is adopted here too, so that several details can be clarified.

Another aim of this paper is to give explicit expressions for  $V$ . We are aware of a substantial literature devoted to the problem of arbitrary unions/intersections of disks/balls [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], yet have not seen anything (in an unsystematic survey) resembling Tao & Wu's results. It is important to remember that our disks *always* possess remote centers and common origin. Thus our results are not general, but nevertheless might constitute an interesting special case for future study.

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# 1. AREAS

We assume WLOG that the  $n$  disk centers are sorted according to increasing argument (counterclockwise angle with respect to the horizontal axis). Since  $V$  is homogeneous and quadratic in  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ , we can reparametrize it as follows:

$$V = r_1^2 V(t_2, t_3, \dots, t_n; \theta_1, \theta_2, \dots, \theta_{n-1})$$

where  $t_k = r_k/r_{k-1}$  and  $\theta_k$  is the (positive) angle between  $\vec{r}_k$  and  $\vec{r}_{k+1}$ . Let  $\theta_n = 2\pi - \theta_1 - \theta_2 - \dots - \theta_{n-1}$ , which is well-defined (positive) by the ordering in our construction. For  $n = 2$ ,

$$V(t_2; \theta_1) = z_1 + t_2^2 z_2$$

where

$$\begin{aligned} z_1 &= \pi - \alpha_1 + \frac{1}{2} \sin(2\alpha_1) - \beta_2 + \frac{1}{2} \sin(2\beta_2), \\ z_2 &= \pi - \alpha_2 + \frac{1}{2} \sin(2\alpha_2) - \beta_1 + \frac{1}{2} \sin(2\beta_1) \end{aligned}$$

and

$$\begin{aligned} \sin(\alpha_1) &= \begin{cases} \frac{t_2 \sin(\theta_1)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_1)}} & \text{if } \theta_1 < \pi, \\ 0 & \text{if } \theta_1 > \pi; \end{cases} \\ \sin(\beta_1) &= \begin{cases} \frac{\sin(\theta_1)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_1)}} & \text{if } \theta_1 < \pi, \\ 0 & \text{if } \theta_1 > \pi; \end{cases} \\ \sin(\alpha_2) &= \begin{cases} \frac{\sin(\theta_2)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_2)}} & \text{if } \theta_2 < \pi, \\ 0 & \text{if } \theta_2 > \pi; \end{cases} \\ \sin(\beta_2) &= \begin{cases} \frac{t_2 \sin(\theta_2)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_2)}} & \text{if } \theta_2 < \pi, \\ 0 & \text{if } \theta_2 > \pi. \end{cases} \end{aligned}$$

The preceding is needlessly complicated. Assume additionally that  $\theta_1 < \pi$  as pictured in Figure 1; it follows that  $\theta_2 = 2\pi - \theta_1 > \pi$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 0$  and therefore

$$z_1 = \pi - \alpha_1 + \frac{1}{2} \sin(2\alpha_1), \quad z_2 = \pi - \beta_1 + \frac{1}{2} \sin(2\beta_1).$$

{Correction to [1]: angle  $\beta_2$  should be  $\beta_1$  in formula (22) and angle  $\beta_{i+1}$  should be  $\beta_i$  in formula (23). The corresponding figure, however, is fine.}

For  $n = 3$ ,

$$V(t_2, t_3; \theta_1, \theta_2) = z_1 + t_2^2 z_2 + t_2^2 t_3^2 z_3$$

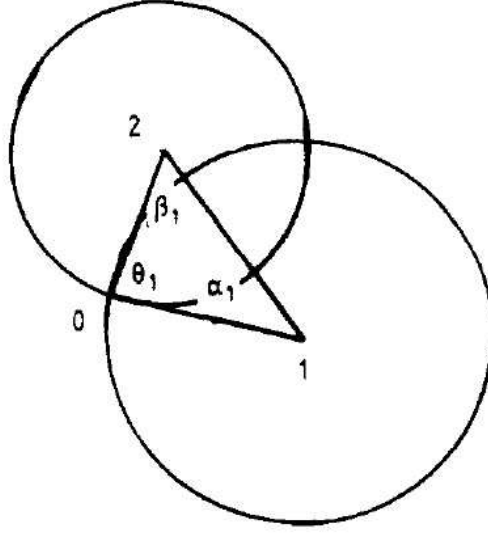


Figure 1: Area occupied by two circles (from [1]).

where

$$\begin{aligned} z_1 &= \pi - \alpha_1 + \frac{1}{2} \sin(2\alpha_1) - \beta_3 + \frac{1}{2} \sin(2\beta_3), \\ z_2 &= \pi - \alpha_2 + \frac{1}{2} \sin(2\alpha_2) - \beta_1 + \frac{1}{2} \sin(2\beta_1), \\ z_3 &= \pi - \alpha_3 + \frac{1}{2} \sin(2\alpha_3) - \beta_2 + \frac{1}{2} \sin(2\beta_2) \end{aligned}$$

and

$$\begin{aligned} \sin(\alpha_1) &= \begin{cases} \frac{t_2 \sin(\theta_1)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_1)}} & \text{if } \theta_1 < \pi, \\ 0 & \text{if } \theta_1 > \pi; \end{cases} \\ \sin(\beta_1) &= \begin{cases} \frac{\sin(\theta_1)}{\sqrt{1 + t_2^2 - 2t_2 \cos(\theta_1)}} & \text{if } \theta_1 < \pi, \\ 0 & \text{if } \theta_1 > \pi; \end{cases} \\ \sin(\alpha_2) &= \begin{cases} \frac{t_3 \sin(\theta_2)}{\sqrt{1 + t_3^2 - 2t_3 \cos(\theta_2)}} & \text{if } \theta_2 < \pi, \\ 0 & \text{if } \theta_2 > \pi; \end{cases} \\ \sin(\beta_2) &= \begin{cases} \frac{\sin(\theta_2)}{\sqrt{1 + t_3^2 - 2t_3 \cos(\theta_2)}} & \text{if } \theta_2 < \pi, \\ 0 & \text{if } \theta_2 > \pi; \end{cases} \end{aligned}$$

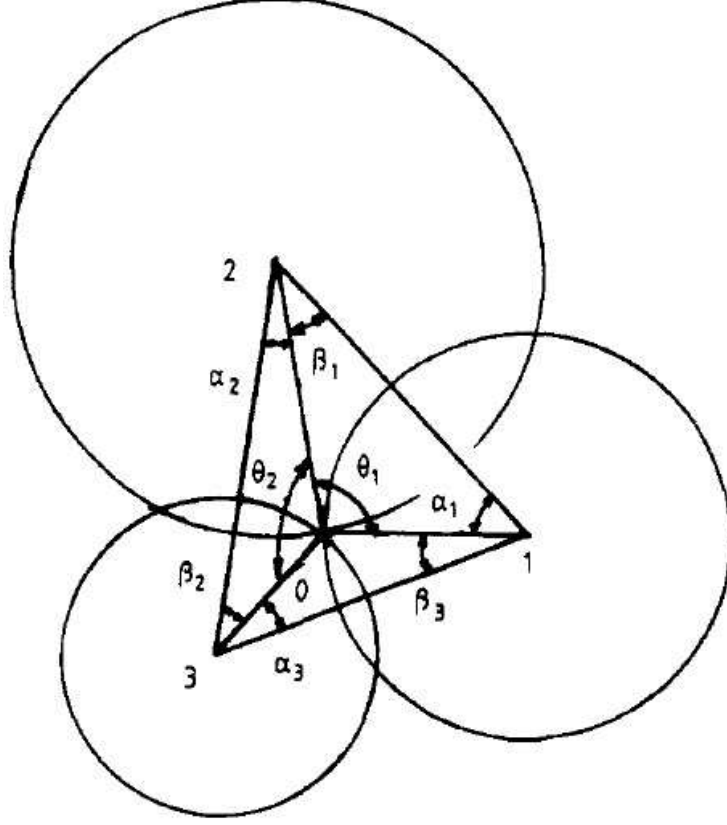


Figure 2: Area occupied by three circles intersecting at a single point (from [1]).

$$\sin(\alpha_3) = \begin{cases} \frac{\sin(\theta_3)}{\sqrt{1 + t_2^2 t_3^2 - 2t_2 t_3 \cos(\theta_3)}} & \text{if } \theta_3 < \pi, \\ 0 & \text{if } \theta_3 > \pi; \end{cases}$$

$$\sin(\beta_3) = \begin{cases} \frac{t_2 t_3 \sin(\theta_3)}{\sqrt{1 + t_2^2 t_3^2 - 2t_2 t_3 \cos(\theta_3)}} & \text{if } \theta_3 < \pi, \\ 0 & \text{if } \theta_3 > \pi. \end{cases}$$

Figure 2 provides a sample scenario in which the overlap is merely the point  $\vec{0}$ ; a more representative picture would include a nondegenerate (circular triangle) intersection. {The conditions for  $\alpha_k$  and  $\beta_k$  to vanish are not fully stated in [1]; we have attempted to be more precise here.}

For arbitrary  $n$ ,

$$V(t_2, t_3, \dots, t_n; \theta_1, \theta_2, \dots, \theta_{n-1}) = z_1 + t_2^2 z_2 + t_2^2 t_3^2 z_3 + \dots + t_2^2 t_3^2 \dots t_n^2 z_n$$

where

$$z_k = \pi - \alpha_k + \frac{1}{2} \sin(2\alpha_k) - \beta_{k-1} + \frac{1}{2} \sin(2\beta_{k-1})$$

for  $1 \leq k \leq n$  and we agree to set  $\beta_0 = \beta_n$ . Also,

$$\sin(\alpha_k) = \begin{cases} \frac{t_{k+1} \sin(\theta_k)}{\sqrt{1 + t_{k+1}^2 - 2t_{k+1} \cos(\theta_k)}} & \text{if } \theta_k < \pi, \\ 0 & \text{if } \theta_k > \pi; \end{cases}$$

$$\sin(\beta_k) = \begin{cases} \frac{\sin(\theta_k)}{\sqrt{1 + t_{k+1}^2 - 2t_{k+1} \cos(\theta_k)}} & \text{if } \theta_k < \pi, \\ 0 & \text{if } \theta_k > \pi \end{cases}$$

for  $1 \leq k \leq n-1$  and

$$\sin(\alpha_n) = \begin{cases} \frac{\sin(\theta_n)}{\sqrt{1 + t_2^2 t_3^2 \cdots t_n^2 - 2t_2 t_3 \cdots t_n \cos(\theta_n)}} & \text{if } \theta_n < \pi, \\ 0 & \text{if } \theta_n > \pi; \end{cases}$$

$$\sin(\beta_n) = \begin{cases} \frac{t_2 t_3 \cdots t_n \sin(\theta_n)}{\sqrt{1 + t_2^2 t_3^2 \cdots t_n^2 - 2t_2 t_3 \cdots t_n \cos(\theta_n)}} & \text{if } \theta_n < \pi, \\ 0 & \text{if } \theta_n > \pi. \end{cases}$$

A proof of these formulas does not appear in [1]. We point out only that, for  $1 \leq k \leq n-1$ ,

$$\frac{\sin(\alpha_k)}{r_{k+1}} = \frac{\sin(\theta_k)}{r_{k,k+1}} = \frac{\sin(\beta_k)}{r_k}$$

by the Law of Sines,

$$r_{k,k+1}^2 = r_k^2 + r_{k+1}^2 - 2r_k r_{k+1} \cos(\theta_k)$$

by the Law of Cosines,  $t_{k+1} = r_{k+1}/r_k$ , hence

$$\frac{\sin(\alpha_k)}{t_{k+1}} = \frac{\sin(\alpha_k)}{r_{k+1}/r_k} = \frac{\sin(\theta_k)}{r_{k,k+1}/r_k} = \sin(\beta_k).$$

Also

$$\frac{\sin(\alpha_n)}{r_1} = \frac{\sin(\theta_n)}{r_{1,n}} = \frac{\sin(\beta_n)}{r_n}$$

by the Law of Sines,

$$r_{1,n}^2 = r_1^2 + r_n^2 - 2r_1 r_n \cos(\theta_n)$$

by the Law of Cosines,  $t_2 t_3 \cdots t_n = r_n/r_1$ , hence

$$\sin(\alpha_n) = \frac{\sin(\theta_n)}{r_{1,n}/r_1} = \frac{\sin(\beta_n)}{r_n/r_1} = \frac{\sin(\beta_n)}{t_2 t_3 \cdots t_n}.$$

## 2. INTEGRALS

Let  $\vec{r}_j = (x_j, y_j)$  for each  $1 \leq j \leq n$  and assume that the argument of  $\vec{r}_1$  is  $\theta_0$ . The change of variables

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} r_1 \cos(\theta_0) \\ r_1 \sin(\theta_0) \\ r_2 \cos(\theta_0 + \theta_1) \\ r_2 \sin(\theta_0 + \theta_1) \\ r_3 \cos(\theta_0 + \theta_1 + \theta_2) \\ r_3 \sin(\theta_0 + \theta_1 + \theta_2) \\ \vdots \\ r_n \cos(\theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{n-1}) \\ r_n \sin(\theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{n-1}) \end{pmatrix} = r_1 \begin{pmatrix} \cos(\theta_0) \\ \sin(\theta_0) \\ t_2 \cos(\theta_0 + \theta_1) \\ t_2 \sin(\theta_0 + \theta_1) \\ t_2 t_3 \cos(\theta_0 + \theta_1 + \theta_2) \\ t_2 t_3 \sin(\theta_0 + \theta_1 + \theta_2) \\ \vdots \\ t_2 t_3 \cdots t_n \cos(\theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{n-1}) \\ t_2 t_3 \cdots t_n \sin(\theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{n-1}) \end{pmatrix}$$

has Jacobian determinant

$$r_1^{2n-1} t_2^{2n-3} t_3^{2n-5} \cdots t_{n-1}^3 t_n.$$

We can reduce the dimensionality of the integral for  $c_n$  by two, using the fact that

$$\int_0^{2\pi} \int_0^\infty r_1^{2n-1} \exp[-r_1^2 V(t_2, t_3, \dots, t_n; \theta_1, \theta_2, \dots, \theta_{n-1})] dr_1 d\theta_0 = \pi(n-1)! V^{-n}.$$

To go forward, it must be understood how  $t_{i+1}$  and  $\theta_i$  interact with each other, as a consequence of  $\max\{r_i, r_{i+1}\} < r_{i,i+1}$ .

Let  $n = 2$  for simplicity's sake. Figure 3 makes clear why  $\pi/3 \leq \theta_1 \leq 5\pi/3$  is a necessary condition for  $\max\{1, t_2^2\} \leq (t_2 \cos(\theta_1) - 1)^2 + t_2^2 \sin(\theta_1)^2$ . A sufficient condition for the latter inequality is

$$\begin{cases} 2 \cos(\theta_1) \leq t_2 \leq \frac{1}{2 \cos(\theta_1)} & \text{if } \frac{\pi}{3} \leq \theta_1 < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < \theta_1 \leq \frac{5\pi}{3}, \\ 0 < t_2 < \infty & \text{if } \frac{\pi}{2} \leq \theta_1 \leq \frac{3\pi}{2} \end{cases}$$

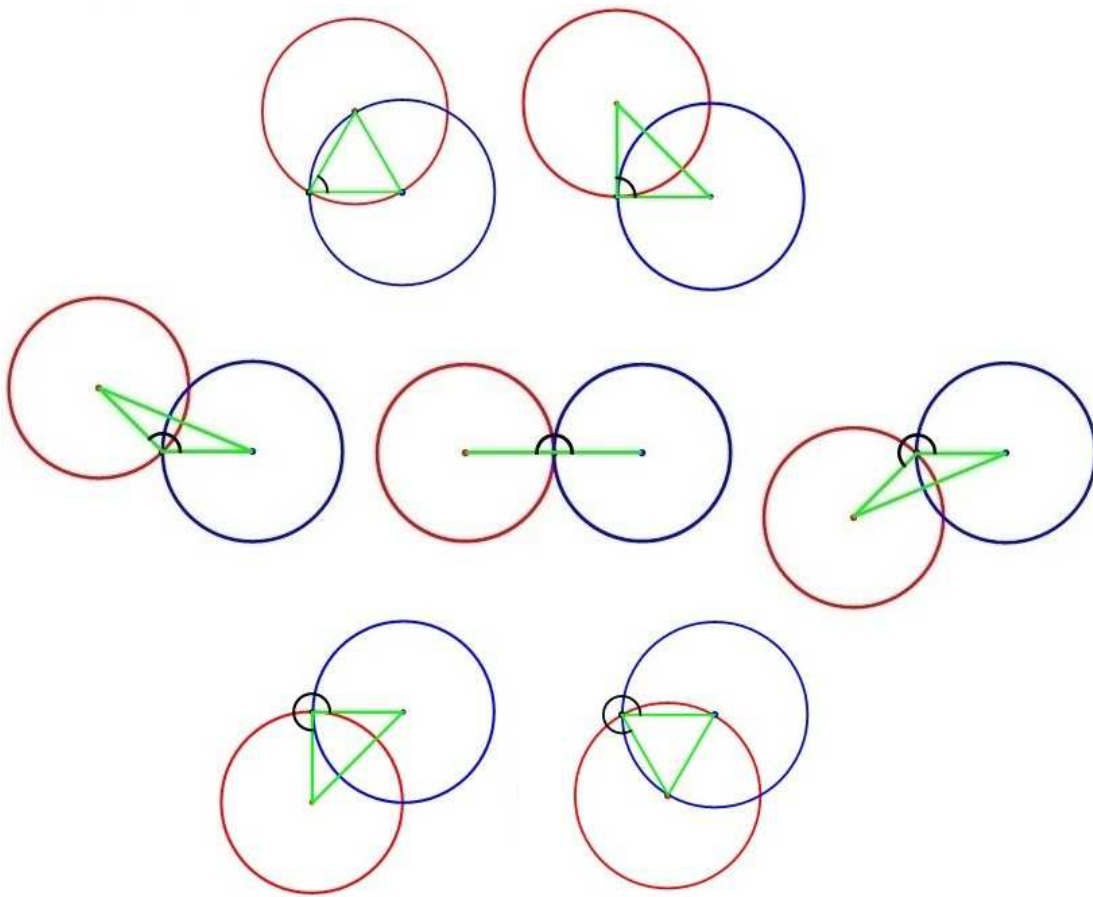


Figure 3: Top row:  $\pi/3 \leq \theta_1 \leq \pi/2$ . Middle row:  $\pi/2 < \theta_1 < 3\pi/2$ . Bottom row:  $3\pi/2 \leq \theta_1 \leq 5\pi/3$ .  $t_2 = 1$  throughout.

and here is a proof. The right hand side reduces to  $t_2^2 - 2t_2 \cos(\theta_1) + 1$ ; when  $t_2 = 2 \cos(\theta_1)$ , this is equal to  $t_2^2 - t_2^2 + 1 = 1$ ; when  $t_2 = 1/(2 \cos(\theta_1))$ , this is equal to  $t_2^2 - 1 + 1 = t_2^2$ . On the one hand, at an interior point  $t_2 = 1$ , the RHS is equal to  $2(1 - \cos(\theta_1))$  and is  $\geq 1$  iff  $\cos(\theta_1) \leq 1/2$ , which is always true in our domain. On the other hand, at a left exterior point  $t_2 = \cos(\theta_1) > 0$ , the RHS is equal to  $1 - \cos(\theta_1)^2$  and is  $< 1$ ; at a right exterior point  $t_2 = 1/\cos(\theta_1) > 0$ , the RHS is equal to  $1/\cos(\theta_1)^2 - 1$  and is  $< t_2^2$ . Finally, if  $\cos(\theta_1) \leq 0$ , then the RHS is  $\geq t_2^2 + 1$  and is easily  $\geq \max\{1, t_2^2\}$ .

A less formal verification of sufficiency is provided in Figures 4 and 5, illustrating the cases when  $\theta_1$  is the midpoint of the interval  $[\pi/3, \pi/2)$  and of  $(3\pi/2, 5\pi/3]$  respectively. A trivial case is  $\theta_1 = \pi$  since the two disks are horizontally tangent at the origin and hence centers are automatically remote, regardless of the value of  $t_2$ .

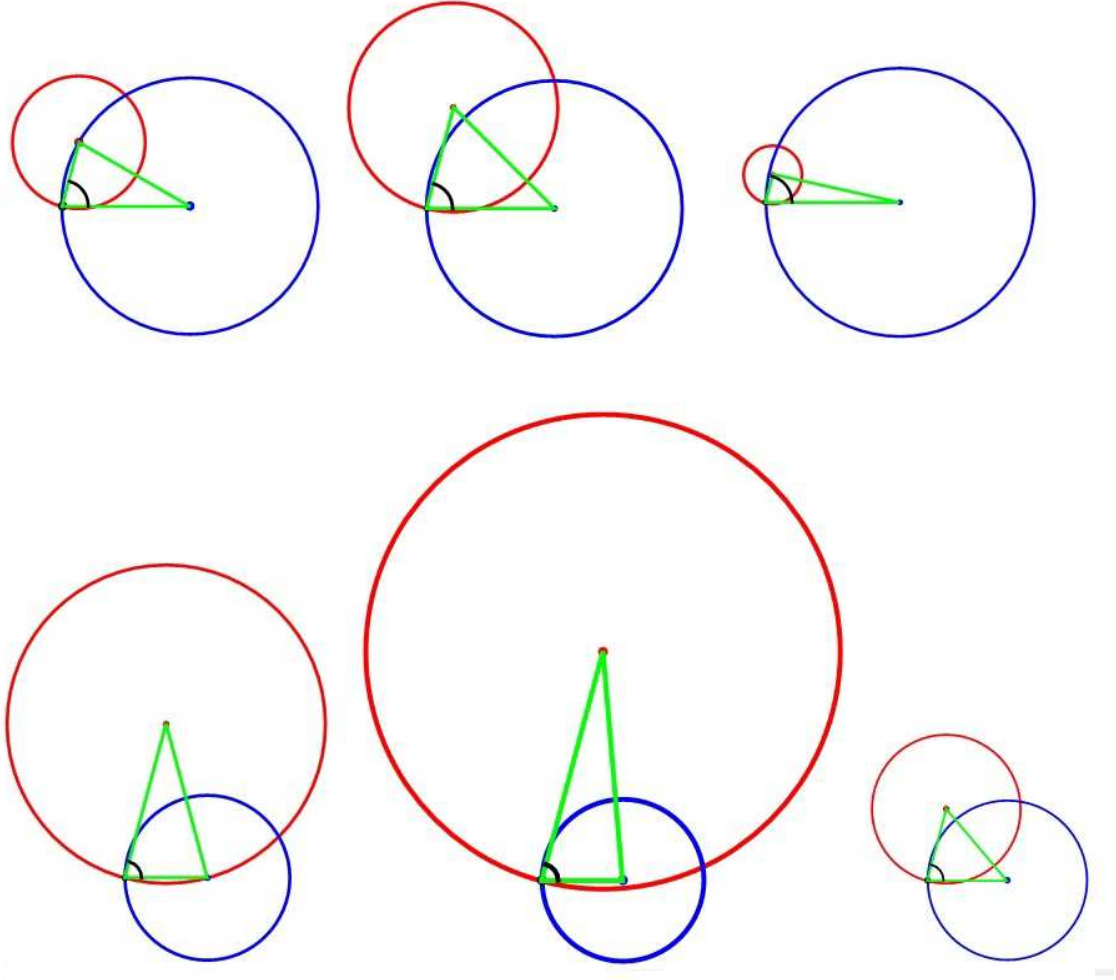


Figure 4: Top row:  $2 \cos(\theta_1)$  is smallest length for which disks have remote centers (violated in third configuration). Bottom row:  $1/(2 \cos(\theta_1))$  is largest length for which disks have remote centers (violated in second configuration).  $\theta_1 = 5\pi/12$  throughout.



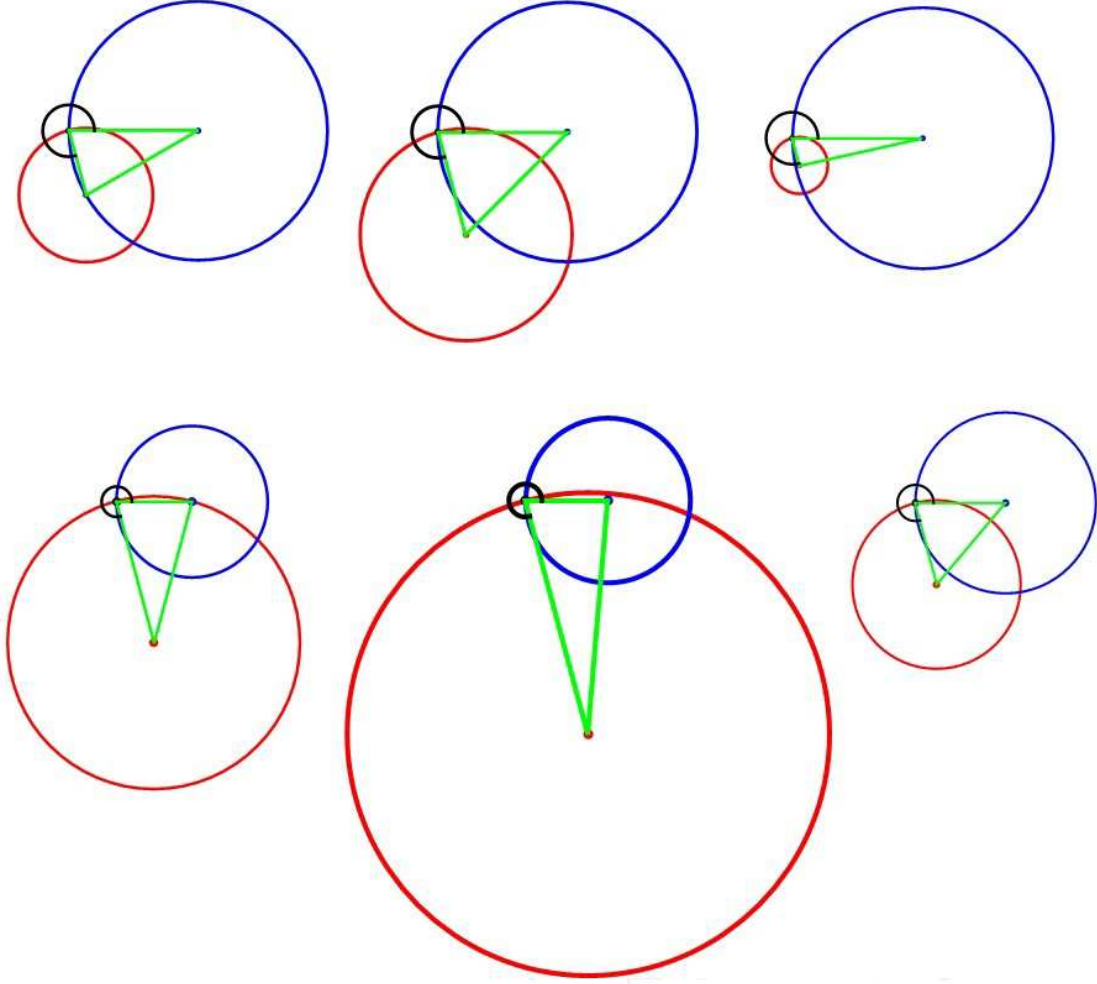


Figure 5: Top row:  $2 \cos(\theta_1)$  is smallest length for which disks have remote centers (violated in third configuration). Bottom row:  $1/(2 \cos(\theta_1))$  is largest length for which disks have remote centers (violated in second configuration).  $\theta_1 = 19\pi/12$  throughout.

We therefore have

$$c_2 = \frac{\pi}{2} \left( \int_{\pi/3}^{\pi/2} \int_{2\cos(\theta_1)}^{1/(2\cos(\theta_1))} + \int_{\pi/2}^{3\pi/2} \int_0^{\infty} + \int_{3\pi/2}^{5\pi/3} \int_{2\cos(\theta_1)}^{1/(2\cos(\theta_1))} \right) \frac{t_2}{V(t_2; \theta_1)^2} dt_2 d\theta_1$$

$$= 0.316585\dots$$

Another representation involves a binary variable  $\sigma_i$  defined as

$$\sigma_i = \begin{cases} 0 & \text{if } \theta_i > \pi/2, \\ 1 & \text{if } \theta_i < \pi/2. \end{cases}$$

Let  $I(\sigma_1, \sigma_2, \dots, \sigma_n)$  be the contribution to  $c_n$  with  $\{\theta_1, \theta_2, \dots, \theta_n\}$  in the range specified by  $\sigma_1, \sigma_2, \dots, \sigma_n$ . Clearly

$$c_2 = I(1, 0) + I(0, 0) + I(0, 1) = I(0, 0) + 2I(1, 0)$$

and, more generally,

$$c_3 = I(0, 0, 0) + 3I(1, 0, 0) + 3I(1, 1, 0),$$

$$c_4 = 4I(0, 0, 0, 0) + 4I(1, 1, 0, 0) + 2I(1, 0, 1, 0) + 4I(1, 1, 1, 0),$$

$$c_5 = 5I(1, 1, 1, 1, 0) + I(1, 1, 1, 1, 1).$$

{The coefficients in formulas (19) and (20) of [1] should be  $\pi/2$ , not  $\pi$ . Tao & Wu's numerical result 0.316333... agrees with ours to three decimal places.} From

$$I(1, 1, 0) = \frac{2\pi}{3} \int_{\pi/3}^{\pi/2} \int_{\pi/3}^{\pi/2} \int_{2\cos(\theta_1)}^{1/(2\cos(\theta_1))} \int_{2\cos(\theta_2)}^{1/(2\cos(\theta_2))} \frac{t_2^3 t_3}{V(t_2, t_3; \theta_1, \theta_2)^3} dt_3 dt_2 d\theta_2 d\theta_1,$$

$$I(1, 0, 0) = \frac{2\pi}{3} \int_{\pi/3}^{\pi/2} \int_{\pi/2}^{3\pi/2-\theta_1} \int_{2\cos(\theta_1)}^{1/(2\cos(\theta_1))} \int_0^{\infty} \frac{t_2^3 t_3}{V(t_2, t_3; \theta_1, \theta_2)^3} dt_3 dt_2 d\theta_2 d\theta_1,$$

$$I(0, 0, 0) = \frac{2\pi}{3} \int_{\pi/2}^{\pi} \int_{\pi/2}^{3\pi/2-\theta_1} \int_0^{\infty} \int_0^{\infty} \frac{t_2^3 t_3}{V(t_2, t_3; \theta_1, \theta_2)^3} dt_3 dt_2 d\theta_2 d\theta_1$$

we obtain  $c_3 = 0.033056\dots$ . Note that  $\theta_3 > \pi/2$  implies that  $\theta_2 = 2\pi - \theta_3 - \theta_1 < 3\pi/2 - \theta_1$ , as indicated for both  $I(1, 0, 0)$  and  $I(0, 0, 0)$ . {Tao & Wu's numerical result 0.032939... agrees with ours to two decimal places.} What is missing, however, is proof that some hitherto undetected interaction between  $t_2$ ,  $t_3$ ,  $\theta_1$ ,  $\theta_2$  does not exist. Figure 6 exhibits underlying complexity; we regrettably must stop here.

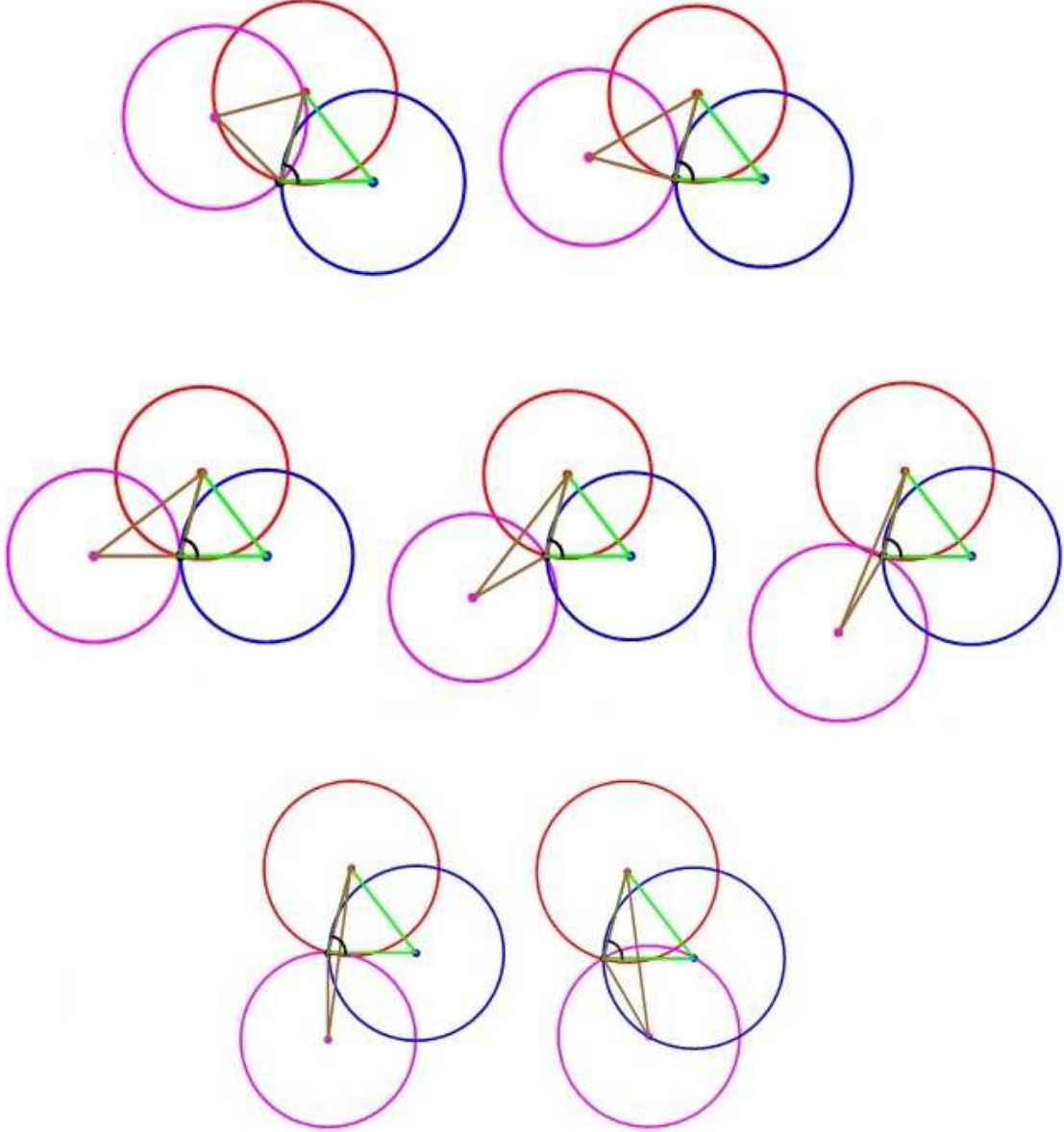


Figure 6: Top row:  $\pi/3 \leq \theta_2 \leq \pi/2$ . Middle row:  $\pi/2 < \theta_2 < 13\pi/12$ . Bottom row:  $13\pi/12 \leq \theta_2 \leq 5\pi/4$ . These correspond to  $I(1, 1, 0)$ ,  $I(1, 0, 0)$ ,  $I(1, 0, 1)$  respectively.  $t_2 = t_3 = 1$  and  $\theta_1 = 5\pi/12$  throughout.

## 3. ACKNOWLEDGEMENTS

I am grateful to Emeritus Professor Fa-Yueh Wu (Northeastern) and Kevin Daily (Wolfram Research) for kind and helpful correspondence. On some future day, if the MATHEMATICA 10 commands

```
NIntegrate[Boole[
x1^2+y1^2 <= (x1-x2)^2+(y1-y2)^2 &&
x2^2+y2^2 <= (x1-x2)^2+(y1-y2)^2] *
Exp[-RegionMeasure[RegionUnion[
Disk[{x1,y1}, Sqrt[x1^2+y1^2]],
Disk[{x2,y2}, Sqrt[x2^2+y2^2]]]],
{x1,-Infinity,Infinity},{y1,-Infinity,Infinity},
{x2,-Infinity,Infinity},{y2,-Infinity,Infinity},
Exclusions -> {x1^2+y1^2 == 0 ||
x2^2+y2^2 == 0}]
```

and

```
NIntegrate[Boole[
x1^2+y1^2 <= (x1-x2)^2+(y1-y2)^2 &&
x2^2+y2^2 <= (x1-x2)^2+(y1-y2)^2 &&
x1^2+y1^2 <= (x1-x3)^2+(y1-y3)^2 &&
x3^2+y3^2 <= (x1-x3)^2+(y1-y3)^2 &&
x2^2+y2^2 <= (x2-x3)^2+(y2-y3)^2 &&
x3^2+y3^2 <= (x2-x3)^2+(y2-y3)^2] *
Exp[-RegionMeasure[RegionUnion[
Disk[{x1,y1}, Sqrt[x1^2+y1^2]],
Disk[{x2,y2}, Sqrt[x2^2+y2^2]],
Disk[{x3,y3}, Sqrt[x3^2+y3^2]]]],
{x1,-Infinity,Infinity},{y1,-Infinity,Infinity},
{x2,-Infinity,Infinity},{y2,-Infinity,Infinity},
{x3,-Infinity,Infinity},{y3,-Infinity,Infinity},
Exclusions -> {x1^2+y1^2 == 0 ||
x2^2+y2^2 == 0 ||
x3^2+y3^2 == 0}]
```

become numerically feasible, then Section 2 of this paper will be rendered unnecessary.

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